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FREE VIBRATIONS OF THIN ELASTIC SPHERICAL SHELLS

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Abstract—This paper is devoted to the problem of free, harmonic vibrations of thin, elastic, spherical shells. The differential equations are rederived in an invariant form together with the appropriate kinematic and static boundary conditions.

The complete solution is presented for axisymmetric and non-axisymmetric vibrational modes for a shell in the shape of a spherical zone with two boundaries. The hitherto unsolved problem of non-axisymmetric vibrations of a spherical shell with a circular opening is included as a special case.

Numerical examples are given covering a variety of boundary conditions and a wide range of the geometrical parameters.

1. INTRODUCTION

The first attempts to solve the theoretical problem of the vibrations of a thin, elastic, spherical shell precede the formulation of the classical theory of shells. Thus, Lamb[1] adapted results derived earlier on the vibrations of an elastic sphere to the case of a shell bounded by two concentric spherical surfaces. By determining the limit as the thickness approaches zero, he obtained the solution for a complete spherical membrane. Lord Rayleigh[2] concluded from physical arguments, that the middle-surface of a vibrating shell remains unstretched and determined the frequencies of an open spherical shell using this condition. Whereas Lamb found that the frequencies were independent of the thickness (extensional vibrations), Lord Rayleigh found them to be directly proportional to the thickness (flexural vibrations). Although Lamb's solution concerned the complete spherical shell and Lord Rayleigh's the open, the two solutions could hardly be reconciled, considering the case of a spherical shell with a small circular opening. Also, it soon became clear, that the conditions at a free boundary could not be satisfied without at least some extension of the middle-surface and that therefore Lord Rayleigh's solution could only be approximate. Its range of validity could not, however, be determined.

In his celebrated paper of 1888, Love[3] combined the effects of flexural and extensional deformations and thus laid the foundations of the now classical theory of shells. He also included in his paper an analysis of the extensional vibrations of open spherical shells.

The general differential equations for spherical shells were derived by van der Neut[4] and Havers[5] for the static case and generalized to the dynamic case by Federhofer[6]. However, the analytical difficulties in solving these equations have until now been overcome only in some special cases, such as shallow shells[7], extensional modes[8], and axisymmetric modes[9], while the general case has not yet received a satisfactory treatment.

In this paper the equations of bending of spherical shells are rederived in an invariant, coordinate-independent form, which is remarkably simple. The complete solution in spherical coordinates is obtained together with static as well as kinematic boundary conditions. The problem of free vibrations leads to an 8×8 frequency determinant equation, and explicit formulas for the elements of this determinant are given.

The work in solving the problem numerically includes the computation of associated Legendre functions of integral order and complex degree, but requires in the whole only a modest effort due to the simple form of the coefficients appearing as elements of the frequency determinant.

2. FIELD EQUATIONS

A spherical surface is characterized by its constant curvature. In mixed form the curvature tensor $d_\beta^{\,\alpha}$ in any coordinate system is equal to plus or minus Kronecker's delta

δ_β^α divided by the radius R of the sphere. The sign will depend on our choice of surface coordinates, and we shall take it such that

$$d_{\alpha\beta} = -\frac{1}{R} a_{\alpha\beta} \tag{2.1}$$

where $a_{\alpha\beta}$ is the metric tensor. With (2.1) the strain tensor takes the form†

$$E_{\alpha\beta} = \frac{1}{2}(D_\alpha v_\beta + D_\beta v_\alpha) + \frac{1}{R} a_{\alpha\beta} w \tag{2.2}$$

and the bending tensor becomes

$$K_{\alpha\beta} = D_\alpha D_\beta w - \frac{1}{R}(D_\alpha v_\alpha + D_\beta v_\alpha) - \frac{1}{R^2} a_{\alpha\beta} w. \tag{2.3}$$

Here, D_α denotes covariant differentiation with respect to the surface coordinate u^α, v_α is the tangential displacement vector, and w the normal displacement.

We shall now take advantage of the possibility of using alternative measures of bending‡. In particular, we see that by taking

$$\tilde{K}_{\alpha\beta} = K_{\alpha\beta} + \frac{2}{R} E_{\alpha\beta} \tag{2.4}$$

the bending tensor becomes a function of w only,

$$\tilde{K}_{\alpha\beta} = D_\alpha D_\beta w + \frac{1}{R^2} a_{\alpha\beta} w \tag{2.5}$$

and that this property will prove useful in leading to simple equations.

The principle of virtual work will require

$$N^{\alpha\beta}\delta E_{\alpha\beta} + M^{\alpha\beta}\delta K_{\alpha\beta} = \tilde{N}^{\alpha\beta}\delta E_{\alpha\beta} + M^{\alpha\beta}\delta\tilde{K}_{\alpha\beta}$$

where $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are the membrane stress and moment tensors, respectively, and $\tilde{N}^{\alpha\beta}$ the augmented membrane stress tensor. The equality will hold good when

$$\tilde{N}^{\alpha\beta} = N^{\alpha\beta} - \frac{2}{R} M^{\alpha\beta} \tag{2.6}$$

which is the appropriate membrane stress tensor.

The equations of equilibrium are§

$$D_\alpha N^{\alpha\beta} + 2d_\gamma^\beta D_\alpha M^{\gamma\alpha} + M^{\gamma\alpha} D_\alpha d_\gamma^\beta + F^\beta = 0 \tag{2.7}$$

and

$$D_\alpha D_\beta M^{\alpha\beta} - d_{\alpha\gamma} d_\beta^\gamma M^{\alpha\beta} - d_{\alpha\beta} N^{\alpha\beta} - p = 0 \tag{2.8}$$

where F^β and p are the external forces.

Substituting (2.1) and (2.6), we find

$$D_\alpha \tilde{N}^{\alpha\beta} + F^\beta = 0 \tag{2.9}$$

†See, e.g. Ref. ([10], pp. 53–62).
‡*Ibid.* pp. 111–113.
§Reference [10], p. 92.

and

$$D_\alpha D_\beta M^{\alpha\beta} + \frac{1}{R^2} M_\alpha^\alpha + \frac{1}{R} \tilde{N}_\alpha^\alpha = p. \tag{2.10}$$

From Hooke's law we get

$$\tilde{N}^{\alpha\beta} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\left(\frac{1}{2}D_\alpha v_\beta + \frac{1}{2}D_\beta v_\alpha + \frac{1}{R}a_{\alpha\beta}w\right) + \nu a_{\alpha\beta}\left(D_\gamma v^\gamma + \frac{2w}{R}\right)\right] \tag{2.11}$$

and

$$M^{\alpha\beta} = D\left[(1-\nu)\left(D^\alpha D^\beta w + \frac{1}{R^2}a^{\alpha\beta}w\right) + \nu a^{\alpha\beta}\left(\Delta + \frac{2}{R^2}\right)w\right] \tag{2.12}$$

where E is Young's modulus, ν Poisson's ratio, h the thickness of the shell and $D = Eh^3/12(1-\nu^2)$ the shell constant. We stress that the uncoupled eqns (2.11) and (2.12) hold good with the same accuracy for $\tilde{K}_{\alpha\beta}$ and $\tilde{N}^{\alpha\beta}$ as for $K_{\alpha\beta}$ and $N^{\alpha\beta}$.

When the shell performs free, harmonic vibrations of small amplitude, the state is governed by the equations of equilibrium (2.9) and (2.10) with F^β and p given by the d'Alembert forces

$$F^\beta = \omega^2\gamma h v^\beta; \quad p = \omega^2\gamma h w \tag{2.13}$$

where ω is the angular frequency and γ the specific mass of the material.

Substitution of (2.11)–(2.13) into (2.9) and (2.10) yields

$$\frac{1-\nu}{2}\left(\Delta + \frac{1}{R^2}\right)v_\beta + \frac{1+\nu}{2}D_\beta D_\gamma v^\gamma + \frac{1+\nu}{R}D_\beta w + \frac{\lambda}{R^2}v_\beta = 0 \tag{2.14}$$

and

$$\left(\Delta + \frac{2}{R^2}\right)\left(\Delta + \frac{1+\nu}{R^2}\right)w + \frac{1+\nu}{kR^3}\left(D_\alpha v^\alpha + \frac{2}{R}w\right) - \frac{\lambda}{kR^4}w = 0 \tag{2.15}$$

respectively, where

$$\lambda = \omega^2(1-\nu^2)\frac{\gamma}{E}R^2 \tag{2.16}$$

and

$$k = \frac{h^2}{12R^2}. \tag{2.17}$$

Since any vector field may be written as the sum of an irrotational and a solenoidal part, we write

$$v_\beta = D_\beta\Psi + \epsilon_{\gamma\beta}D^\gamma\chi \tag{2.18}$$

where Ψ and χ are two scalar functions. Taking the covariant derivative and contracting, we get

$$D_\beta v^\beta = \Delta\Psi. \tag{2.19}$$

If instead of contracting, we multiply by the alternating tensor $\epsilon^{\alpha\beta}$, we find

$$\epsilon^{\alpha\beta}D_\alpha v_\beta = \Delta\chi \tag{2.20}$$

which is the local average rotation.

By substituting (2.14) and observing the rules for interchanging the order of covariant differentiation,† we get

$$D_\beta\left(\Delta\Psi+\frac{1-\nu}{R^2}\Psi+\frac{\lambda}{R^2}\Psi+\frac{1+\nu}{R}w\right)+\frac{1-\nu}{2}\epsilon_{\gamma\beta}D^\gamma\left(\Delta\chi+\frac{2}{R^2}\chi+\frac{2\lambda}{(1-\nu)R^2}\chi\right)=0. \tag{2.21}$$

Multiplying by the operator $a^{\alpha\beta}D_\alpha$, we get

$$\Delta\left(\Delta\Psi+\frac{1-\nu}{R^2}\Psi+\frac{\lambda}{R^2}\Psi+\frac{1+\nu}{R}w\right)=0 \tag{2.22}$$

and hence

$$\left(\Delta+\frac{1-\nu}{R^2}+\frac{\lambda}{R^2}\right)\Psi+\frac{1+\nu}{R}w=\kappa \tag{2.23}$$

where κ is a harmonic function.

Multiplying (2.21) instead by $\epsilon^{\alpha\beta}D_\alpha$, we get

$$\Delta\left(\Delta+\frac{2}{R^2}+\frac{2\lambda}{(1-\nu)R^2}\right)\chi=0 \tag{2.24}$$

and thus

$$\left(\Delta+\frac{2}{R^2}+\frac{2\lambda}{(1-\nu)R^2}\right)\chi=\mu \tag{2.25}$$

where μ is a harmonic function.

By substituting (2.23) and (2.25) back into (2.21) we find

$$D_\beta\kappa+\frac{1-\nu}{2}\epsilon_{\gamma\beta}D^\gamma\mu=0 \tag{2.26}$$

which, when compared with the Cauchy–Riemann equations for the real and imaginary parts of an analytic function, shows that κ and $(1-\nu)/2\mu$ are conjugate harmonic.

A particular integral of (2.23) is

$$\Psi_0=\frac{k}{\dfrac{1-\nu}{R^2}+\dfrac{\lambda}{R^2}} \tag{2.27}$$

and of (2.25)

$$\chi_0=\frac{\mu}{\dfrac{2}{R^2}+\dfrac{2\lambda}{(1-\nu)R^2}}. \tag{2.28}$$

Substituting Ψ_0 and χ_0 into (2.18), we get

$$v_\beta=\frac{D_\beta\kappa}{\dfrac{1-\nu}{R^2}+\dfrac{\lambda}{R^2}}+\epsilon_{\gamma\beta}\frac{D^\gamma\mu}{\dfrac{2}{R^2}+\dfrac{2\lambda}{(1-\nu)R^2}}=0$$

due to (2.26). Thus, conjugate harmonic functions do not contribute to the displacements, and we may therefore, without loss of generality, take $\kappa=\mu=0$.

†See Appendix A.

The first two equations of equilibrium can therefore be written in terms of the rotation and dilatation as

$$\left(\Delta + \frac{2}{R^2} + \frac{2\lambda}{(1-\nu)R^2}\right)\chi = 0 \tag{2.29}$$

and

$$\left(\Delta + \frac{1-\nu}{R^2} + \frac{\lambda}{R^2}\right)\Psi + \frac{1+\nu}{R}w = 0 \tag{2.30}$$

respectively. The third equation is obtained when (2.18) is substituted into (2.15),

$$\left(\Delta + \frac{2}{R^2}\right)\left(\Delta + \frac{1+\nu}{R^2}\right)w + \frac{1+\nu}{kR^3}\left(\Delta\Psi + \frac{2}{R}w\right) - \frac{\lambda}{kR^4}w = 0. \tag{2.31}$$

The equations of equilibrium (2.29)–(2.31) are thus expressed in terms of three scalar functions, χ, Ψ, and w. The only differential operator appearing in the equations is the invariant Laplacian operator Δ.

The equations are partly decoupled, since only χ appears in the first one, and since it only appears there. In general, however, coupling is caused by the boundary conditions.

It should be stressed, that the equations are accurate, in the sense that they are fully consistent with Love's first approximation.

3. BOUNDARY CONDITIONS

Let δv_α and δw be arbitrary virtual displacements and $\delta E_{\alpha\beta}$, $\delta K_{\alpha\beta}$ the corresponding tensors of strain and bending according to (2.2) and (2.3). The forces and moments acting at the boundary, corresponding to the field tensors $N^{\alpha\beta}$ and $M^{\alpha\beta}$, are derived from the principle of virtual work, which leads to the following equality†

$$\iint_{\mathscr{D}} (N^{\alpha\beta}\delta E_{\alpha\beta} + M^{\alpha\beta}\delta K_{\alpha\beta})\,\mathrm{d}A = \oint_{\mathscr{C}} \left(T^\alpha \delta v_\alpha + Q\delta w + M_B \frac{\partial}{\partial n}\delta w\right)\mathrm{d}s \tag{3.1}$$

where the surface integral is extended over a domain $\mathscr{D}$ of the middle-surface and the line integral over the closed boundary $\mathscr{C}$. Here

$$T^\alpha = (N^{\beta\alpha} + 2d_\gamma^{\ \alpha} M^{\beta\gamma})n_\beta \tag{3.2}$$

is the membrane force vector,

$$Q = -n_\beta D_\alpha M^{\alpha\beta} - \frac{\partial}{\partial s}(M^{\alpha\beta} n_\alpha t_\beta) \tag{3.3}$$

is the effective shear force,

$$M_B = M^{\alpha\beta} n_\alpha n_\beta \tag{3.4}$$

the bending moment, and n_α, t_α the unit normal and tangent vectors, respectively, to the curve $\mathscr{C}$.

Using (2.4) and (2.6) the equality (3.1) can be written in the form

$$\iint_{\mathscr{D}} (\tilde{N}^{\alpha\beta}\delta E_{\alpha\beta} + M^{\alpha\beta}\delta \tilde{K}_{\alpha\beta})\,\mathrm{d}A = \oint_{\mathscr{C}} \left(N\delta u + S\delta v + Q\delta w + M_B \frac{\partial}{\partial n}\delta w\right)\mathrm{d}s \tag{3.5}$$

†Reference [10], p. 91.

where u and v are the normal and tangential displacements at the boundary,

$$N = \tilde{N}^{\alpha\beta} n_\alpha n_\beta \tag{3.6}$$

and

$$S = \tilde{N}^{\alpha\beta} n_\alpha t_\beta \tag{3.7}$$

are the normal and tangential membrane stresses, respectively.

With the help of (2.11) and (2.12) we find

$$N = \frac{Eh}{1-\nu^2}\left[(1-\nu)(n^\alpha n^\beta D_\alpha D_\beta \Psi - n^\alpha t^\beta D_\alpha D_\beta \chi) + \nu \Delta \Psi + \frac{1+\nu}{R} w\right] \tag{3.8}$$

$$S = \frac{Eh}{2(1+\nu)}[2n^\alpha t^\beta D_\alpha D_\beta \Psi + (n^\alpha n^\beta - t^\alpha t^\beta) D_\alpha D_\beta \chi] \tag{3.9}$$

$$M_B = D\left[(1-\nu) n^\alpha n^\beta D_\alpha D_\beta w + \nu \Delta w + \frac{1+\nu}{2} w\right] \tag{3.10}$$

$$Q = -D(1-\nu)\left[n^\alpha D_\alpha\left(\Delta + \frac{2}{R^2}\right) w + t^\alpha D_\alpha (n^\beta t^\gamma D_\beta D_\gamma w)\right]. \tag{3.11}$$

The bending moment M_B and the three boundary forces N, S, and Q are at our disposal and can be freely prescribed along the boundary, provided that they are in equilibrium with the external (d'Alembert) forces. The four boundary conditions and the eight-order system (2.29)–(2.31) constitute a well-posed mathematical problem.

The static boundary conditions may be replaced by kinematic conditions according to the r.h.s. of (3.5). Thus, we may prescribe

either N or $u = \dfrac{\partial \Psi}{\partial n} - \dfrac{\partial \chi}{\partial s}$, and

either S or $v = \dfrac{\partial \Psi}{\partial s} + \dfrac{\partial \chi}{\partial n}$, and

either M_B or $\dfrac{\partial w}{\partial n}$, and

either Q or w.

or, more generally, four linear combinations of these quantities.

Whether we have static or kinematic conditions at the boundary, we see, that in general, there will be a coupling between the function χ and the two functions Ψ and w through the boundary conditions. However, if the shell is complete, there are no boundary conditions and therefore no such coupling. In that case χ can be determined independently of ψ and w, and *vice versa*.

4. SOLUTION OF THE FIELD EQUATIONS

By retaining the Laplacian Δ as an operator, and observing that all coefficients of (2.30) and (2.31) are constant, we can solve the system by an operator method. To do that, we write (2.30) and (2.31) in the form

$$\left.\begin{aligned} A_{11} w + A_{12} \Psi = 0 \\ A_{21} w + A_{22} \Psi = 0 \end{aligned}\right\} \tag{4.1}$$

where the operators are given by

$$\left.\begin{aligned} A_{11} &= \frac{1+\nu}{R} & A_{12} &= \Delta + \frac{1-\nu}{R^2} + \frac{\lambda}{R^2} \\ A_{21} &= \left(\Delta + \frac{2}{R^2}\right)\left(\Delta + \frac{1+\nu}{R^2}\right) + \frac{2(1+\nu)}{kR^4} - \frac{\lambda}{kR^4} & A_{22} &= \frac{1+\nu}{kR^3}\Delta \end{aligned}\right\} \tag{4.2}$$

Taking

$$w = -A_{12}\Xi; \quad \Psi = A_{11}\Xi \tag{4.3}$$

the system (4.1) will be satisfied whenever Ξ is a solution of the equation

$$(A_{11}A_{22} - A_{12}A_{21})\Xi = 0 \tag{4.4}$$

where Ξ is a *potential function* for solving the system (4.1) for spherical shells.

We can factorize this differential equation and write it in the form

$$\left(\Delta - \frac{\beta}{R^2}\right)\left(\Delta - \frac{\xi + i\eta}{R^2}\right)\left(\Delta - \frac{\xi - i\eta}{R^2}\right)\Xi = 0 \tag{4.5}$$

where β is the real root, and $\xi \pm i\eta$ the two complex conjugate roots of the following algebraic equation of third degree in Δ,

$$\Delta^3 + c_2\Delta^2 + c_1\Delta + c_0 = 0 \tag{4.6}$$

where

$$\left.\begin{aligned} c_2 &= 4 + \lambda \\ c_1 &= (1 - \nu^2 - \lambda)/k + 5 - \nu^2 + (3+\nu)\lambda \\ c_0 &= [2(1-\nu^2) + (1+3\nu)\lambda - \lambda^2]/k + 2(1-\nu^2) + 2(1+\nu)\lambda \end{aligned}\right\} \tag{4.7}$$

For given values of ν, k and λ, this equation can be solved by *Cardan's formula* or other well-known methods.

For small values of λ, β is close to -2, ξ close to -1, and η approximately equal to $\sqrt{(1-\nu^2)/k}$. Thus, η is of order R/h, while β and ξ are of order unity. The small terms in (4.7) are therefore important and cannot be omitted without seriously affecting the result.

We find, that the complete solution of the problem can be expressed in term of the solution to one single second-order differential equation. Using conventional notation we shall write that in the form

$$\left[\Delta + \frac{\sigma(\sigma+1)}{R^2}\right]y = 0 \tag{4.8}$$

where σ is a real number in two of the equations and complex conjugate in the remaining two. When λ is greater than a certain critical value λ_c, which for all values of k and ν of practical interest lies in the neighbourhood of one ($\lambda_c \approx 1$), all three roots of (4.6) become real (*casus irreducibilis*), and the parameter σ will be real in all four equations. This makes the analysis and the numerical evaluation neither simpler nor more complicated, although naturally, the mere fact that the two cases $\lambda < \lambda_c$ and $\lambda > \lambda_c$ have to be treated separately, since they require different algorithms, is a complication.

In the following we shall give the analysis and all relevant formulas for the important case $\lambda < \lambda_c$, while for simplicity, the corresponding formulas for $\lambda > \lambda_c$ will be omitted.

To proceed, we introduce spherical coordinates θ, ϕ, where θ is the co-latitude, and ϕ the longitude. We shall also restrict our analysis to solutions that are periodic in ϕ, with the period 2π. This will include complete shells, shells with one boundary $\theta = \alpha$ ("bowls" or "bells"), and shells with two boundaries $\theta = \alpha_1$ and $\theta = \alpha_2$ (spherical "zones"). In all cases we can expand the field functions in Fourier series. For the solution y we write

$$y = \sum_{m=0}^{\infty} [C_m(\theta)\cos m\phi + D_m(\theta)\sin m\phi] \tag{4.9}$$

where $C_m(\theta)$ and $D_m(\theta)$ are functions of θ.

When (4.9) is substituted into (4.8) and the coefficients of the trigonometric functions are put equal to zero, we get

$$\left[\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta} + \sigma(\sigma+1) - \frac{m^2}{\sin^2\theta}\right]C_m(\theta) = 0 \tag{4.10}$$

for which the complete solution is†

$$C_m(\theta) = AP_\sigma^{-m}(\cos\theta) + BP_\sigma^{-m}(\cos(\pi - \theta)) \tag{4.11}$$

where P_σ^{-m} is the associated Legendre function of the first kind, degree σ, and order $-m$. A and B are arbitrary constants.

Since σ may be complex, the functions are in general complex, and we shall write

$$P_\sigma^{-m} = R_\sigma^{-m} + iS_\sigma^{-m} \tag{4.12}$$

where R_σ^{-m} and S_σ^{-m} are the real and imaginary parts, respectively.

The complete solution is given by

$$\left.\begin{aligned} \Xi &= \Big[A_1P_\sigma^{-m}(\cos\theta) + A_2R_\zeta^{-m}(\cos\theta) + A_3S_\zeta^{-m}(\cos\theta) \\ &\quad + A_5P_\sigma^{-m}(\cos(\pi-\theta)) + A_6R_\sigma^{-m}(\cos(\pi-\theta)) \\ &\quad + A_7S_\zeta^{-m}(\cos(\pi-\theta))\Big]\cos m\phi \\ \chi &= \frac{1}{R}[A_4P_\tau^{-m}(\cos\theta) + A_8P_\tau^{-m}(\cos(\pi-\theta))]\sin m\phi \end{aligned}\right\} \tag{4.13}$$

where

$$\sigma(\sigma+1) = -\beta; \quad \zeta(\zeta+1) = -\xi - i\eta; \quad \tau(\tau+1) = -\gamma. \tag{4.14}$$

The parameters β, ξ, and η are given by the roots of eqn (4.6), and γ is determined by the condition that χ satisfies (2.29), i.e.

$$\gamma = 2 + \frac{2\lambda}{1-\nu}. \tag{4.15}$$

With the potentials Ξ and χ determined, we proceed to find all relevant quantities. For that purpose we shall need the derivatives of R_σ^{-m} and S_σ^{-m}.

From the formula‡

$$(1-x^2)\frac{d}{dx}P_\sigma^{-m} = -\sqrt{1-x^2}P_\sigma^{-m+1} + mxP_\sigma^{-m} \tag{4.16}$$

†See appendix B.
‡See Ref. [11], p. 114.

we find

$$\left.\begin{aligned} \frac{d}{d\theta} R_\sigma^{-m} &= R_\sigma^{-m+1} - m \cot\theta R_\sigma^{-m} \\ \frac{d}{d\theta} S_\sigma^{-m} &= S_\sigma^{-m+1} - m \cot\theta R_\sigma^{-m} \end{aligned}\right\}. \tag{4.17}$$

Also, if $(R_\sigma^{-m} + iS_\sigma^{-m})\cos m\phi$ satisfies the differential equation

$$\left(\Delta + \frac{a+ib}{R^2}\right)(R_\sigma^{-m} + iS_\sigma^{-m})\cos m\phi = 0 \tag{4.18}$$

with $a + ib = \sigma(\sigma + 1)$, we deduce that

$$\left.\begin{aligned} R_\sigma^{-m+2} &= 2(m-1)\cot\theta R_\sigma^{-m+1} - (a - m^2 + m)R_\sigma^{-m} + bS_\sigma^{-m} \\ S_\sigma^{-m+2} &= 2(m-1)\cot\theta S_\sigma^{-m+1} - (a - m^2 + m)S_\sigma^{-m} - bS_\sigma^{-m} \end{aligned}\right\}. \tag{4.19}$$

The second and all higher derivatives can therefore be expressed in terms of functions of order $-m$ and $-m+1$.

The normal displacement w and the scalar function Ψ are obtained when Ξ according to (4.13) is substituted into (4.3). With χ, Ψ, and w in hand we can determine all relevant quantities: displacements, forces, and moments.

Substitution into (3.8)–(3.11) yields

$$\left.\begin{aligned} N &= \frac{Eh}{R^3}\left(\sum_{i=1}^{8} c_{Ni}A_i\right)\cos m\phi \\ S &= \frac{Eh}{R^3}\left(\sum_{i=1}^{8} c_{Si}A_i\right)\sin m\phi \\ M_B &= \frac{D}{R^4}\left(\sum_{i=1}^{8} c_{Mi}A_i\right)\cos m\phi \\ Q &= \frac{D}{R^5}\left(\sum_{i=1}^{8} c_{Qi}A_i\right)\cos m\phi \\ u &= \frac{1}{R^2}\left(\sum_{i=1}^{8} c_{ui}A_i\right)\cos m\phi \\ v &= \frac{1}{R^2}\left(\sum_{i=1}^{8} c_{vi}A_i\right)\sin m\phi \\ w &= \frac{1}{R^2}\left(\sum_{i=1}^{8} c_{wi}A_i\right)\cos m\phi \\ \frac{\partial w}{\partial n} &= \frac{1}{R^3}\left(\sum_{i=1}^{8} c_{\psi i}A_i\right)\cos m\phi. \end{aligned}\right\} \tag{4.20}$$

The 64 coefficients $c_{Ni}, \ldots, c_{\psi 8}$ can be written more compactly by using the following abbreviations,

$$\left.\begin{aligned}
a_1 &= \frac{m(m+1)}{\sin^2\theta}\\
a_2 &= (m+1)\left(\frac{m}{\sin^2\theta}-1\right)-\frac{\lambda}{1-\nu}\\
a_3 &= (1-\nu)(m+1)\left(\frac{m}{\sin^2\theta}-1\right)+2-\beta\\
a_4 &= \lambda+1-\nu-\beta\\
a_5 &= (1-\nu)(m+1)\left(\frac{m}{\sin^2\theta}-1\right)+2-\xi\\
a_6 &= \lambda+1-\nu-\xi\\
a_7 &= (1-\nu)\frac{m(m+1)}{\sin^2\theta}-2\\
a_8 &= (1-\nu)\frac{m^2}{\sin^2\theta}-2
\end{aligned}\right\} \tag{4.21}$$

and

$$\left.\begin{aligned}
\mathscr{P} &= P_\sigma^{-m}(\cos\theta); & \hat{\mathscr{P}} &= P_\sigma^{-m}(\cos(\pi-\theta))\\
\mathscr{P}_1 &= P_\sigma^{-m+1}(\cos\theta); & \hat{\mathscr{P}}_1 &= P_\sigma^{-m+1}(\cos(\pi-\theta))\\
\mathscr{Q} &= P_\tau^{-m}(\cos\theta); & \hat{\mathscr{Q}} &= P_\tau^{-m}(\cos(\pi-\theta))\\
\mathscr{Q}_1 &= P_\tau^{-m+1}(\cos\theta); & \hat{\mathscr{Q}}_1 &= P_\tau^{-m+1}(\cos(\pi-\theta))\\
\mathscr{R} &= R_\zeta^{-m}(\cos\theta); & \hat{\mathscr{R}} &= R_\zeta^{-m}(\cos(\pi-\theta))\\
\mathscr{R}_1 &= R_\zeta^{-m+1}(\cos\theta); & \hat{\mathscr{R}}_1 &= R_\zeta^{-m+1}(\cos(\pi-\theta))\\
\mathscr{S} &= S_\zeta^{-m}(\cos\theta); & \hat{\mathscr{S}} &= S_\zeta^{-m}(\cos(\pi-\theta))\\
\mathscr{S}_1 &= S_\zeta^{-m+1}(\cos\theta); & \hat{\mathscr{S}}_1 &= S_\zeta^{-m+1}(\cos(\pi-\theta)).
\end{aligned}\right\} \tag{4.22}$$

We get

$$\left.\begin{aligned}
c_{N1} &= a_2\mathscr{P}-\cot\theta\,\mathscr{P}_1\\
c_{N2} &= a_2\mathscr{R}-\cot\theta\,\mathscr{R}_1\\
c_{N3} &= a_2\mathscr{S}-\cot\theta\,\mathscr{S}_1\\
c_{N4} &= \frac{m}{(1+\nu)\sin\theta}[(m+1)\cot\theta\,\mathscr{Q}-\mathscr{Q}_1]\\
c_{N5} &= a_2\hat{\mathscr{P}}+\cot\theta\,\hat{\mathscr{P}}_1\\
c_{N6} &= a_2\hat{\mathscr{R}}+\cot\theta\,\hat{\mathscr{R}}_1\\
c_{N7} &= a_2\hat{\mathscr{S}}+\cot\theta\,\hat{\mathscr{S}}_1\\
c_{N8} &= \frac{m}{(1+\nu)\sin\theta}[(m+1)\cot\theta\,\hat{\mathscr{Q}}+\hat{\mathscr{Q}}_1]
\end{aligned}\right\} \tag{4.23}$$

$$\left.\begin{aligned}
c_{S1} &= a_1 \cos\theta\, \mathscr{P} - \frac{m}{\sin\theta}\, \mathscr{P}_1 \\
c_{S2} &= a_1 \cos\theta\, \mathscr{R} - \frac{m}{\sin\theta}\, \mathscr{R}_1 \\
c_{S3} &= a_1 \cos\theta\, \mathscr{S} - \frac{m}{\sin\theta}\, \mathscr{S}_1 \\
c_{S4} &= \left(a_1 - m - \frac{\gamma}{2}\right)\frac{\mathscr{Q}}{1+\nu} - \frac{\cot\theta}{1+\nu}\, \mathscr{Q}_1 \\
c_{S5} &= a_1 \cos\theta\, \hat{\mathscr{P}} + \frac{m}{\sin\theta}\, \hat{\mathscr{P}}_1 \\
c_{S6} &= a_1 \cos\theta\, \hat{\mathscr{R}} + \frac{m}{\sin\theta}\, \hat{\mathscr{R}}_1 \\
c_{S7} &= a_1 \cos\theta\, \hat{\mathscr{S}} + \frac{m}{\sin\theta}\, \hat{\mathscr{S}}_1 \\
c_{S8} &= \left(a_1 - m - \frac{\gamma}{2}\right)\frac{\hat{\mathscr{Q}}}{1+\nu} + \frac{\cot\theta}{1+\nu}\, \hat{\mathscr{Q}}_1
\end{aligned}\right\} \tag{4.24}$$

$$\left.\begin{aligned}
c_{M1} &= a_4[(1-\nu)\cot\theta\, \mathscr{P}_1 - a_3\mathscr{P}] \\
c_{M2} &= (\eta^2 - a_5a_6)\mathscr{R} - \eta(a_5+a_6)\mathscr{S} + (1-\nu)\cot\theta\,(a_6\mathscr{R}_1 + \eta\mathscr{S}_1) \\
c_{M3} &= (\eta^2 - a_5a_6)\mathscr{S} + \eta(a_5+a_6)\mathscr{R} + (1-\nu)\cot\theta(a_6\mathscr{S}_1 - \eta\mathscr{R}_1) \\
c_{M4} &= 0 \\
c_{M5} &= -\, a_4[(1-\nu)\cot\theta\hat{\mathscr{P}}_1 + a_3\hat{\mathscr{P}}] \\
c_{M6} &= (\eta^2 - a_5a_6)\hat{\mathscr{R}} - \eta(a_5+a_6)\hat{\mathscr{S}} - (1-\nu)\cot\theta(a_6\hat{\mathscr{R}}_1 + \eta\hat{\mathscr{S}}_1) \\
c_{M7} &= (\eta^2 - a_5a_6)\hat{\mathscr{S}} + \eta(a_5+a_6)\hat{\mathscr{R}} - (1-\nu)\cot\theta(a_6\hat{\mathscr{S}}_1 - \eta\hat{\mathscr{R}}_1) \\
c_{M8} &= 0
\end{aligned}\right\} \tag{4.25}$$

$$\left.\begin{aligned}
c_{Q1} &= m\cot\theta a_4(a_7+\beta)\mathscr{P} - a_4(a_8+\beta)\mathscr{P}_1 \\
c_{Q2} &= m\cot\theta\eta(a_7 - a_6 + \xi)\mathscr{S} + m\cot\theta[\eta^2 + a_6(a_7+\xi)]\mathscr{R} \\
&\quad - [\eta^2 + a_6(a_8+\xi)]\mathscr{R}_1 - \eta(a_8 - a_6 + \xi)\mathscr{S}_1 \\
c_{Q3} &= m\cot\theta\eta(a_7 - a_6 + \xi)\mathscr{R} + m\cot\theta[\eta^2 + a_6(a_7+\xi)]\mathscr{S} \\
&\quad - [\eta^2 + a_6(a_8+\xi)]\mathscr{S}_1 + \eta(a_8 - a_6 + \xi)\mathscr{R}_1 \\
c_{Q4} &= 0 \\
c_{Q5} &= m\cot\theta a_4(a_7+\beta)\hat{\mathscr{P}} + a_4(a_8+\beta)\hat{P}_1 \\
c_{Q6} &= m\cot\theta\eta(a_7 - a_6 + \xi)\hat{\mathscr{S}} + m\cot\theta[\eta^2 + a_6(a_7+\xi)]\hat{\mathscr{R}} \\
&\quad + [\eta^2 + a_6(a_8+\xi)]\hat{\mathscr{R}}_1 + \eta(a_8 - a_6 + \xi)\hat{\mathscr{S}}_1 \\
c_{Q7} &= -m\cot\theta\eta(a_7 - a_6 + \xi)\hat{\mathscr{R}} + m\cot\theta[\eta^2 + a_6(a_7+\xi)]\hat{\mathscr{S}} \\
&\quad - [\eta^2 + a_6(a_8+\xi)]\hat{\mathscr{S}}_1 - \eta(a_8 - a_6 + \xi)\hat{\mathscr{R}}_1 \\
c_{Q8} &= 0
\end{aligned}\right\} \tag{4.26}$$

$$\left.\begin{aligned}
c_{u1} &= (1+\nu)(\mathscr{P}_1 - m\cot\theta\,\mathscr{P})\\
c_{u2} &= (1+\nu)(\mathscr{R}_1 - m\cot\theta\,\mathscr{R})\\
c_{u3} &= (1+\nu)(\mathscr{S}_1 - m\cot\theta\,\mathscr{S})\\
c_{u4} &= -\frac{m}{\sin\theta}\,\mathscr{Q}\\
c_{u5} &= (1+\nu)(\hat{\mathscr{P}}_1 + m\cot\theta\,\hat{\mathscr{P}})\\
c_{u6} &= (1+\nu)(\hat{\mathscr{R}}_1 + m\cot\theta\,\hat{\mathscr{R}})\\
c_{u7} &= (1+\nu)(\hat{\mathscr{S}}_1 + m\cot\theta\,\hat{\mathscr{S}})\\
c_{u8} &= \frac{m}{\sin\theta}\,\hat{\mathscr{Q}}
\end{aligned}\right\} \tag{4.27}$$

$$\left.\begin{aligned}
c_{v1} &= -(1+\nu)\frac{m}{\sin\theta}\,\mathscr{P}\\
c_{v2} &= -(1+\nu)\frac{m}{\sin\theta}\,\mathscr{R}\\
c_{v3} &= -(1+\nu)\frac{m}{\sin\theta}\,\mathscr{S}\\
c_{v4} &= -m\cot\theta\,\mathscr{Q} + \mathscr{Q}_1\\
c_{v5} &= (1+\nu)\frac{m}{\sin\theta}\,\hat{\mathscr{P}}\\
c_{v6} &= (1+\nu)\frac{m}{\sin\theta}\,\hat{\mathscr{R}}\\
c_{v7} &= (1+\nu)\frac{m}{\sin\theta}\,\hat{\mathscr{S}}\\
c_{v8} &= m\cot\theta\,\hat{\mathscr{Q}} + \mathscr{Q}_1
\end{aligned}\right\} \tag{4.28}$$

$$\left.\begin{aligned}
c_{w1} &= -a_4\mathscr{P}\\
c_{w2} &= -a_6\mathscr{R} - \eta\,\mathscr{S}\\
c_{w3} &= -a_6\mathscr{S} + \eta\,\mathscr{R}\\
c_{w4} &= 0\\
c_{w5} &= a_4\hat{\mathscr{P}}\\
c_{w6} &= a_6\hat{\mathscr{R}} + \eta\,\hat{\mathscr{S}}\\
c_{w7} &= a_6\hat{\mathscr{S}} - \eta\,\hat{\mathscr{R}}\\
c_{w8} &= 0
\end{aligned}\right\} \tag{4.29}$$

$$\left.\begin{aligned}
c_{\psi 1} &= a_4(m \cot\theta\mathscr{P} - \mathscr{P}_1)\\
c_{\psi 2} &= a_6(m \cot\theta\mathscr{R} - \mathscr{R}_1) + m \cot\theta\eta\mathscr{S} - \eta\mathscr{S}_1\\
c_{\psi 3} &= a_6(m \cot\theta\mathscr{S} - \mathscr{S}_1) - m \cot\theta\eta\mathscr{R} + \eta\mathscr{R}_1\\
c_{\psi 4} &= 0\\
c_{\psi 5} &= -\, a_4(m \cot\theta\hat{\mathscr{P}} + \hat{\mathscr{P}}_1)\\
c_{\psi 6} &= -\, a_6(m \cot\theta\hat{\mathscr{R}} + \hat{\mathscr{R}}_1) - m \cot\theta\eta\hat{\mathscr{S}} - \eta\hat{\mathscr{S}}_1\\
c_{\psi 7} &= -\, a_6(m \cot\theta\hat{\mathscr{S}} + \hat{\mathscr{S}}_1) + m \cot\theta\eta\hat{\mathscr{R}} + \eta\hat{\mathscr{R}}_1\\
c_{\psi 8} &= 0.
\end{aligned}\right\} \tag{4.30}$$

We are now prepared to formulate a solution to the problem. For a shell in the shape of a spherical zone there are four homogeneous boundary conditions at each boundary. This leads by (4.20) to a linear, homogeneous system of eight equations for the eight unknown coefficients $A_1, \ldots, A_8$. The condition for a non-trivial solution is that the determinant vanishes. For instance, if both boundaries are free, we get

$$\begin{vmatrix}
c_{N1}(\alpha_1)\ldots c_{N8}(\alpha_1)\\
c_{S1}(\alpha_1)\ldots c_{S8}(\alpha_1)\\
c_{M1}(\alpha_1)\ldots c_{M8}(\alpha_1)\\
c_{Q1}(\alpha_1)\ldots c_{Q8}(\alpha_1)\\
c_{N1}(\alpha_2)\ldots c_{N8}(\alpha_2)\\
c_{S1}(\alpha_2)\ldots c_{S8}(\alpha_2)\\
c_{M1}(\alpha_2)\ldots c_{M8}(\alpha_2)\\
c_{Q1}(\alpha_2)\ldots c_{Q8}(\alpha_2)
\end{vmatrix} = 0 \tag{4.31}$$

The roots of this transcendental equation determine the natural frequencies of a spherical shell with two free boundaries.

When σ is not an integer, the associated Legendre functions $P_\sigma^{-m}(x)$ have a singularity at $x = 1$. Therefore, if the shell contains the pole $\theta = 0$, the coefficients A_5, A_6, A_7, and A_8 must be zero. This reduces the number of unknown coefficients to four (A_1, A_2, A_3, A_4), and that is precisely the number needed to satisfy four conditions at one boundary $\theta = \alpha$. For a spherical bowl with a free boundary the frequency determinant is just

$$\begin{vmatrix}
c_{N1} & c_{N2} & c_{N3} & c_{N4}\\
c_{S1} & c_{S2} & c_{S3} & c_{S4}\\
c_{M1} & c_{M2} & c_{M3} & c_{M4}\\
c_{Q1} & c_{Q2} & c_{Q3} & c_{Q4}
\end{vmatrix} = 0 \tag{4.32}$$

where all elements are evaluated at $\theta = \alpha$.

When the shell is complete, both poles $\theta = 0$ and $\theta = \pi$ are included, and there remains no possible solution, unless σ is integral ($\sigma = n$).

Also, when the shell is complete, the first equation of equilibrium (2.29) is decoupled from the remaining two. A solution of the system is therefore

$$\chi = P_n^{-m}(\cos\theta)\cos m\phi; \quad w \equiv \Psi \equiv 0 \tag{4.33}$$

which, when substituted into (2.29)–(2.31) leads to the relation

$$n(n+1) = 2 + \frac{2\lambda}{1-\nu}.$$

Utilizing (2.16) we find the corresponding frequencies to be

$$\omega = \sqrt{(n-1)(n+2)}\,\frac{1}{R}\sqrt{\frac{G}{\gamma}} \qquad n = 1, 2, \ldots \tag{4.34}$$

where $G = E/2(1+\nu)$ is the shear modulus.

For each integer $n > 1$ there is one axisymmetric mode and n non-axisymmetric modes, given by $m = 1, 2, \ldots, n$, corresponding to the surface harmonics of order m. The frequency ω however, is independent of m. This seemingly paradoxical situation can readily be explained as a consequence of spherical symmetry. Thus, for instance, the mode $\chi = P_2^{-2}(\cos\theta)\cos 2\phi$ can be obtained by superimposing two axisymmetric modes $P_2(\cos\theta)$, one tilted 90° relative to the other, and with amplitudes in the ratio 1:2. In fact, all higher modes of a given degree n can be obtained by superimposing a number of axisymmetric modes of degree n, with properly chosen axes and amplitudes. Therefore, the frequency depends on n only, not on m.

Besides (4.33), the equations of equilibrium (2.29)–(2.31) are also solved by

$$\chi \equiv 0; \quad w = AP_n^{-m}(\cos\theta)\cos m\phi; \quad \Psi = BRP_n^{-m}(\cos\theta)\cos m\phi. \tag{4.35}$$

Substitution leads to the following linear system

$$\left.\begin{aligned} (1+\nu)A + [1-\nu+\lambda-n(n+1)]B = 0 \\ \left\{[2-n(n+1)][1+\nu-n(n+1)] + \frac{2(1+\nu)-\lambda}{k}\right\}A - \frac{1+\nu}{k}n(n+1)B = 0 \end{aligned}\right\}. \tag{4.36}$$

When $n = 0$, the second equation yields

$$\lambda = 2(1+\nu)(1+k)$$

and hence, after omitting the small term k,

$$\omega^2 = \frac{2E}{(1-\nu)\gamma R^2}. \tag{4.37}$$

The mode is $P_0 \equiv 1$, corresponding to a uniform extension-compression of the shell in a wholly radial motion.

In the general case ($\nu > 0$) the frequency is determined from the condition that the determinant of (4.36) vanishes. For thin shells and low numbers n, n^4 can be neglected in comparison with $1/k$. In such cases we find the equation

$$\lambda^2 - \lambda(n^2+n+1+3\nu) + (1-\nu^2)(n^2+n-2) = 0. \tag{4.38}$$

For each value of n there are two roots to this equation, corresponding to modes of quite different character.

Equations equivalent to (4.34) and (4.38) were derived by Lamb[1] but in a different way. He found that the frequencies were independent of h, which is characteristic of purely extensional vibrations. Of course, all modes (4.35) are contaminated with bending, and if the "small" terms of (4.36) are retained, we shall find that λ is in fact dependent on h/R.

5. NUMERICAL RESULTS

We shall find it convenient to give the frequency in the form

$$\omega = c\frac{h}{R^2}\sqrt{\frac{G}{\gamma}}. \tag{5.1}$$

Table 1. First frequency of a spherical shell with one or two free boundaries α_1 and α_2 for $m = 2$, $h/R = 0.01$, and $\nu = 0.3$

Coefficient c

α_1	α_2					
	0°	15°	30°	45°	60°	75°
15°	39.0787					
30°	10.4706	8.5815				
45°	5.0520	4.7600	3.6433			
60°	3.1843	3.1070	2.7983	2.1798		
75°	2.3929	2.3669	2.2593	1.9989	1.6133	
90°	2.0809	2.0711	2.0275	1.8942	1.6622	1.4015
105°	2.0923	2.0884	2.0676	1.9800	1.7730	1.5679
120°	2.4846	2.4830	2.4689	2.3700	2.0523	1.7730
135°	3.6599	3.6590	3.6369	3.2486	2.3700	1.9800
150°	7.4287	7.4267	6.9705	3.3669	2.4689	2.0676
165°	27.1373	26.5061	7.4267	3.6590	2.4830	2.0884
180°	(118.493)	27.1373	7.4287	3.6599	2.4846	2.0923

Here, c is a dimensionless coefficient, which for given boundary conditions will depend on five parameters. Three of them, h/R, α_1, and α_2 describe the geometry of the shell, one is Poisson's ratio ν, and finally there is the wavenumber m.

For inextensional vibrations—if such existed—c would be independent of h/R and ν. For purely extensional vibrations (such do exist), c is inversely proportional to h/R. Our choice of representation (5.1) reflects the fact that all modes connected with the lower range of the spectrum are mainly flexural, i.e. in this range c depends only weakly on h/R and ν.

For given values of the five parameters, the frequency determinant (4.31) has an infinite number of roots, and we shall refer to them and their modes in order of magnitude, calling the lowest the first, etc. From a practical point of view only the first few are of interest.

In working out the numerical values, the roots of the frequency determinant were calculated by a simple iterative method to an accuracy corresponding to eight significant

Table 2. Second frequency of a spherical shell with two free boundaries α_1 and α_2 for $m = 2$, $h/R = 0.01$, and $\nu = 0.3$

Coefficient c

α_1	α_2				
	15°	30°	45°	60°	75°
30°	47.2592				
45°	33.8057	14.2289			
60°	30.2816	10.3504	6.6661		
75°	28.8240	8.8893	5.2616	4.2597	
90°	28.0970	8.2272	4.5658	3.6505	3.4573
105°	27.6984	7.8886	4.2151	3.3233	3.2483
120°	27.4665	7.7039	4.0581	3.2862	3.3233
135°	27.3253	7.6088	4.2652	4.0581	4.2151
150°	27.2379	7.9891	7.6088	7.7039	7.8886
165°	27.8199	27.2379	27.3253	27.4665	27.6984

Table 3. First and second frequency of an equatorial zone for $h/R = 0.04$ and $\nu = 0.3$

Coefficient c

α_1	α_2	First two modes	
		In plane	Out of plane
100°	80°	1.3046	3.2050
95°	85°	1.2553	2.9794
92°	88°	1.2496	1.9889
91°	89°	1.2491	1.1488
Ring		1.2490	—

figures. The associated Legendre functions appearing in the formulas were calculated using the hypergeometric series as explained in Appendix B.

The coefficient c for the first frequency in the case of a free spherical shell with one or two boundaries and with $h/R = 0.01$, $\nu = 0.3$ and $m = 2$ is given in Table 1 as a function of the co-latitudes of the boundaries α_1 and $\alpha_2(\alpha_1 > \alpha_2 \geq 0)$.

Since bending for thin shells is confined to a comparatively narrow region at the boundary, the frequency is strongly dependent on the size of the larger opening but only weakly on the size of the smaller opening. Naturally, this does not hold when the intermediate zone is so narrow, that the bending regions at the two boundaries interact.

The second frequency (Table 2) on the other hand is more or less determined by the smaller opening.

In the first mode, the sign of w is the same at both boundaries for the same longitude, and the amplitude is largest at the larger opening. The reverse is the case in the second mode, the deflection being of opposite sign at the boundaries, and the amplitude largest at the smaller opening.

When a spherical shell in the shape of an equatorial zone becomes sufficiently narrow, it will behave like a slender circular ring. It is well known, that flexural vibrations of a

Table 4. First frequency of a spherical shell with one or two free boundaries α_1 and α_2 for $\alpha_1 = 90^\circ$, $h/R = 0.01$, and $\nu = 0.3$

Coefficient c

m	α_2					
	0°	15°	30°	45°	60°	75°
2	2.0809	2.0711	2.0275	1.8942	1.6622	1.4015
3	5.7238	5.7233	5.7068	5.5252	4.8570	3.9829
4	10.8168	10.8168	10.8128	10.6860	9.6110	7.6830
5	17.1903	17.1903	17.1895	17.1278	15.9049	12.5063

Table 5. First frequency of a spherical shell with two free boundaries α_1 and α_2 for $\alpha_1 = 90^\circ$, $m = 2$, and $\nu = 0.3$

Coefficient c

h/R	α_2					
	0°	15°	30°	45°	60°	75°
0.005	2.0997	2.0903	2.0492	1.9217	1.6973	1.4743
0.01	2.0809	2.0711	2.0275	1.8942	1.6622	1.4015
0.02	2.0520	2.0415	1.9943	1.8525	1.6058	1.3287
0.04	2.0073	1.9957	1.9427	1.7852	1.5114	1.2936

Table 6. First frequency of a spherical shell with two free boundaries α_1 and α_2 for $\alpha_1 = 90°$, $h/R = 0.01$ and $m = 2$

Coefficient c

ν	α_2 0°	15°	30°	45°	60°	75°
0.0	2.0582	2.0477	2.0011	1.8602	1.6172	1.3152
0.1	2.0661	2.0558	2.0103	1.8720	1.6328	1.3456
0.2	2.0736	2.0636	2.0191	1.8833	1.6478	1.3742
0.3	2.0809	2.0711	2.0275	1.8942	1.6622	1.4015
0.4	2.0881	2.0784	2.0359	1.9049	1.6764	1.4281
0.5	2.0952	2.0857	2.0441	1.9155	1.6904	1.4542

circular ring fall into two classes, i.e. flexural vibrations in the plane of the ring and flexural vibrations involving displacements at right angles to the plane of the ring as well as twist.† The two classes are recognized as the first and second mode, described above. For a zone of diminishing width the two frequencies are given in Table 3. Comparison with the result for a ring shows excellent agreement.

When the angular width of the zone is sufficiently small, the moment of inertia of the cross section with respect to an axis in the equatorial plane becomes smaller than moment of inertia with respect to an axis perpendicular to this plane. The frequency for out-of-plane vibrations may then be the lowest one. This is clearly the case for $h/R = 0.04$ and 2°.

It may also be checked that the value obtained for a very shallow spherical bowl with $\alpha_1 = 15°$ and $\alpha_2 = 0°$ ($c = 39.0787$ in Table 1) is in good agreement with the value $c = 39.031$ obtained for a free circular plate of the same diameter and thickness.

For the higher wave numbers $m = 3, 4, \ldots$ etc. the results are quite analogous to those for $m = 2$. However, since the bending region actually narrows down for increasing wave numbers, the interaction between the boundaries becomes even less pronounced. This is illustrated by Table 4, which gives the dimensionless frequency c for a hemispherical shell ($\alpha_1 = 90°$) with an opening of varying size for $m = 2, 3, 4$, and 5. It may be seen that the influence of the size of the smaller opening diminishes with increasing m.

Table 7. Second frequency of spherical shells with one free boundary for $h/R = 0.01$, and $\nu = 0.3$

Coefficient c

α	m = 0	m = 1	m = 2
30°	157.973	162.805	
45°	153.087	156.685	160.149
60°	148.331	152.139	155.841
75°	144.110	147.055	152.045
90°	140.363	141.324	147.929
105°	136.489	135.289	143.334
120°	132.173	129.520	138.303
135°	127.626	124.727	133.018
150°	123.321	121.279	127.654
165°	119.929	119.219	122.496

†Reference [12], pp. 452–453.

Table 8. First frequency of an open spherical shell with $\nu = 0.3$

Coefficient c

α	m	h/R				Rayleigh
		0.04	0.02	0.01	0.005	
60°	2	3.085	3.145	3.184	3.210	3.264
	3	7.647	7.929	8.132	8.267	8.537
	4	13.659	14.350	14.901	15.284	16.050
90°	2	2.007	2.052	2.081	2.100	2.139
	3	5.356	5.577	5.724	5.819	6.012
	4	9.834	10.411	10.817	11.086	11.619
120°	2	2.341	2.429	2.485	2.521	2.596
	3	6.608	7.025	7.300	7.477	7.830
	4	12.191	13.239	13.974	14.461	15.413
150°	2	6.315	6.989	7.429	7.710	8.257
	3	16.811	19.479	21.372	22.645	25.094
	4	28.538	34.489	38.917	42.076	48.377

As already mentioned mentioned, c would be independent of h/R and ν, if the modes were inextensional. Since no mode is wholly inextensional, however, c will depend on h/R and ν, as we may see in Tables 5 and 6. They show the coefficient c for the first mode of a hemispherical shell with an opening of varying size, and for a wide range of thickness ratios h/R and Poisson's ratio ν. The variation in c amounts to a few per cent, but increases as the width of the spherical zone narrows, just as we would expect.

The modes for the wave numbers $m = 0$ and $m = 1$ are fundamentally different from those of the higher wave numbers. In both cases the first root of the frequency determinant is zero and the corresponding modes describe rigid body motion. Since there is not even a kinematical possibility for inextensional deformation at $m = 0$ and $m = 1$, all such modes involve substantial extension of the middle surface and the frequencies are considerably higher than those of the flexural modes, being comparable in magnitude to those given by Lamb for the complete spherical shell (Table 7). Also, the bending region is much narrower than for $m = 2$, and the frequency is practically independent of the size of the smaller opening, expect for very narrow zones.

When $\alpha_2 = 0$ the shell has but one boundary $\theta = \alpha$, and the frequency determinant is

Table 9. First frequency of spherical shells with one boundary for $m = 2$, $h/R = 0.01$, and $\nu = 0.3$

Coefficient c

α	Boundary conditions				
	1	2	3	4	5
60°	3.184	70.062	127.321	156.774	157.170
90°	2.081	51.925	121.616	145.198	145.324
120°	2.485	43.864	122.703	132.664	132.743
150°	7.429	51.252	121.622	123.405	123.547

No.	Static	Kinematic
1	$N = S = M_B = Q = 0$	
2	$N = S = M_B = 0$	$w = 0$
3	$S = M_B = 0$	$u = w = 0$
4	$M_B = 0$	$u = v = w = 0$
5		$u = v = w = \frac{dw}{dn} = 0$

reduced to a 4×4 array, which is given by (4.32) for a free boundary. In Table 8 the coefficients c are given for different values of α, m, and h/R. They may be compared with Lord Rayleigh's solution to the problem, which was based on the assumption of inextensional deformation of the middle-surface. The agreement is good for low values of α, $m (m \geq 2)$, and h/R. For values of α approaching 180° the errors become increasingly grave. Naturally, we would expect this, since it is well known that the frequencies according to Lord Rayleigh's theory tend to infinity as α approaches 180°. At first, it may look more surprising that the agreement for a hemispherical shell for example—although good—is not much better. It has been argued† that extension of the middle-surface, although necessarily present for satisfying the boundary conditions, is practically confined to a narrow boundary region. But it seems that two factors have been overlook here. Firstly, not only the boundary conditions, but also the field equations, require at least some extension of the middle-surface, but this is probably not so important. Secondly, however, and this is more important, bending is also confined to a narrow region near the edge, and it is the ratio of strain energy to bending energy that matters. We believe that this fully explains the difference between Lord Rayleigh's approximation and the results of the present analysis.

The influence of different boundary conditions are illustrated in Table 9, which gives the first frequency of an open spherical shell with one boundary. As the static boundary conditions are replaced, one by one, by the corresponding kinematic conditions, the frequency is raised. It is evident that the largest increment is achieved in the first steps, replacing $Q = 0$ by $w = 0$, and $N = 0$ by $u = 0$. This is due to the fact that the condition $u = w = 0$ renders an inextensible spherical surface completely rigid‡ and that consequently, such a condition will make a thin spherical shell very stiff. The modes are largely extensional, and hence the last step, replacing $M_B = 0$ by $\mathrm{d}w/\mathrm{d}n = 0$, has an almost negligible influence on the frequency.

Finally, it should be pointed out that the high accuracy with which the coefficient c has been given in all tables above, should not be misinterpreted. naturally, c or rather c^2 is a quantity which, in a shell theory based on Love's first approximation, will be affected by a relative error of order h/R. The many digits in the tables are given for the sole purpose of demonstrating the influence of the different parameters on the frequency.

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APPENDIX A

The Riemann–Christoffel tensor for the spherical surface is§

$$B_{\sigma\alpha\beta\gamma} = \frac{1}{R^2}(a_{\sigma\gamma}a_{\alpha\beta} - a_{\sigma\beta}a_{\alpha\gamma}) \tag{A1}$$

†See Ref. [12], p. 552.

‡This is based on the following theorem by Jellett [13]: *If any curve be traced upon an inextensible surface, whose principal curvatures are finite and of the same sign, and if this curve be rendered immovable, the entire surface will become immovable also.*

§Reference [10], p. 37.

and the rules for interchanging the order of covariant differentiation of a vector and a second order tensor are found to be

$$D_\alpha D_\beta A^\gamma - D_\beta D_\alpha A^\gamma = \frac{1}{R^2}(\delta_\alpha^\gamma A_\beta - \delta_\beta^\gamma A_\alpha) \tag{A2}$$

and

$$D_\alpha D_\beta A_{\gamma\delta} - D_\beta D_\alpha A_{\gamma\delta} = \frac{1}{R^2}(a_{\alpha\gamma} A_{\beta\delta} - a_{\gamma\beta} A_{\alpha\delta} + a_{\delta\alpha} A_{\gamma\beta} - a_{\delta\beta} A_{\gamma\alpha}). \tag{A3}$$

Thus, we have

$$\Delta D_\beta \Psi = D_\alpha D^\alpha D_\beta \Psi = D_\alpha D_\beta D^\alpha \Psi = D_\beta\left(\Delta + \frac{1}{R^2}\right)\Psi \tag{A4}$$

and

$$D_\alpha D_\beta D^\alpha D^\beta w = \Delta^2 w + \frac{1}{R^2}\Delta w. \tag{A5}$$

APPENDIX B

The associated Legendre functions of the first kind "on the cut", i.e. for real values of $x(|x| < 1)$, are defined by†

$$P_\sigma^{-m}(x) = \frac{1}{\Gamma(m+1)}\left(\frac{1-x}{1+x}\right)^{m/2} F\left(-\sigma, \sigma+1, 1+m, \frac{1-x}{2}\right) \tag{B1}$$

where m is an arbitrary non-negative integer, σ arbitrary (complex), and F the hypergeometric function, defined by the series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1!\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \cdots \quad (|x| < 1). \tag{B2}$$

When α and β are complex, each term is complex, and we shall write $a_n + ib_n$ for the nth term. Then

$$\frac{a_n + ib_n}{a_{n-1} + ib_{n-1}} = \frac{(\alpha+n)(\beta+n)x}{(n+1)(\gamma+n)} = \frac{n^2 + n(\alpha+\beta) + \alpha\beta}{(n+1)(n+\gamma)}x \tag{B3}$$

and hence, when we replace $\alpha\beta$ by $-\sigma(\sigma+1)$, i.e. by $-\xi - i\eta$, $\alpha + \beta$ by 1, γ by $m+1$, and x by $(1-x)/2$, we get

$$\left.\begin{aligned} R_\sigma^{-m}(x) &= \frac{1}{m!}\left(\frac{1-x}{1+x}\right)^{m/2}\left(1 + \sum_{n=0}^{\infty} a_n\right) \\ S_\sigma^{-m}(x) &= \frac{1}{m!}\left(\frac{1-x}{1+x}\right)^{m/2}\left(\sum_{n=0}^{\infty} b_n\right) \end{aligned}\right\} \tag{B4}$$

where

$$a_0 = -\frac{\xi(1-x)}{2(m+1)}; \quad b_0 = -\frac{\eta(1-x)}{2(m+1)} \tag{B5}$$

and

$$\left.\begin{aligned} a_n &= [(n^2 + n - \xi)a_{n-1} + \eta b_{n-1}]\frac{1-x}{2(n+1)(n+m+1)} \\ b_n &= [(n^2 + n - \xi)b_{n-1} + \eta a_{n-1}]\frac{1-x}{2(n+1)(n+m+1)} \end{aligned}\right\} \tag{B6}$$

These formulas were used for numerical evaluation of the Legendre functions in the present analysis for $m > 0$. For $m < 0$ the hypergeometric function in (B1) breaks down. In that case we take‡

$$P_\sigma^{-m}(x) = (-1)^m \frac{\Gamma(\sigma - m + 1)}{\Gamma(\sigma - m + 1)} P_\sigma^m(x). \tag{B7}$$

For the complete solution we also need the associated Legendre functions of the second kind $Q_\sigma^m(x)$, for which we have the relation§

†Reference [11], p. 143.
‡Reference [11], p. 140.
§*Ibid.* p. 144.

$$P_\sigma^m(-x) = P_\sigma^m(x)\cos[\pi(\sigma+m)] - \frac{2}{\pi}Q_\sigma^m(x)\sin[\pi(\sigma+m)]. \tag{B8}$$

Since σ is not an integer, but m is, the coefficient for Q_σ^m does not vanish, and hence $Q_\sigma^m(x)$ can be written as a linear combination of $P_\sigma^m(x)$ and $P_\sigma^m(-x)$. The functions $P_\sigma^m(x)$ and $P_\sigma^m(-x)$ are therefore two linearly independent solutions of (4.10), and hence we have the complete solution.